

Breathing mode compactifications and supersymmetry of the brane-world

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ABSTRACT

It has recently been shown that the Randall-Sundrum brane-world may be obtained from an appropriate doubled D3-brane configuration in type IIB theory. This corresponds, in five dimensions, to a sphere compactification of the original IIB theory with a non-trivial breathing mode supporting the brane. In this paper, we shall study the supersymmetry of this reduction to massive five-dimensional supergravity, and derive the effective supersymmetry transformations for the fermionic superpartners to the breathing mode. We also consider the sphere compactifications of eleven-dimensional supergravity to both four and seven dimensions. For the compactifications on S^5 and S^7 , we include a squashing mode scalar and discuss the truncation from $N = 8$ to $N = 2$ supersymmetry.

1 Introduction

The Randall-Sundrum brane-world idea has attracted much recent attention as a mechanism for trapping gravity with an uncompact extra dimension. Although this idea leads to interesting new phenomenology and cosmology, for a while it was not clear how fundamental this idea was, as it did not seem to fit into any established framework for quantum gravity. Along these lines, even the supersymmetry of the scenario itself was in doubt. In particular, general no-go theorems were proven indicating that Randall-Sundrum type domain wall solutions cannot be obtained in the context of $D = 5$, $N = 2$ supergravity [1, 2]. Additionally, it was shown in [3] that the Randall-Sundrum brane tension is greater than that expected from a corresponding stack of D3-branes, thus hinting that the brane-world is necessarily non-BPS.

Despite these difficulties with supersymmetry, it was suggested in [4, 5, 6, 7, 8] that the warped geometry of the brane-world lends itself to a natural interpretation in the framework of Maldacena's AdS/CFT correspondence. In this context, the Randall-Sundrum brane may be viewed as a boundary imposed on a given horospherical slice of AdS_5 . Since this is no longer a boundary at infinity, gravity no longer decouples, and the correspondence is thus one between AdS_5 with a boundary and a cutoff CFT coupled to gravity. This provides an alternate explanation for the trapping of gravity by the brane.

The picture of the Randall-Sundrum mechanism as a generalized AdS/CFT conjecture was given further support in [9], where the one-loop corrected graviton propagator calculated in the $N = 4$ SCFT was shown to agree with the classical graviton propagator obtained for the Randall-Sundrum AdS geometry. While not a direct test of the supersymmetry of the brane-world, it nevertheless provided further evidence for at least compatibility of the brane-world with supersymmetry.

Despite the tension question of [3], progress has also been made in directly relating the Randall-Sundrum model to compactifications of ten-dimensional type IIB theory in the presence of D3-branes [10, 11, 12, 13]. These compactifications have the feature of retaining the breathing mode scalar [14], to be used in supporting the five-dimensional domain wall solution. It has subsequently been realized in [15] that it is precisely the breathing mode scalar that allows the evasion of the supergravity no-go theorems mentioned above. In particular, since the breathing mode is a massive mode, it lies outside the context of [1, 2], which only considers scalars in massless $N = 2$ vector multiples.

Additionally, the realization of the brane-world in [15] corresponds to a doubled D3-brane geometry in ten dimensions, where the Randall-Sundrum brane is in fact a composite of a negative tension D3-brane and an orbifold plane. This structure of the Randall-Sundrum brane both stabilizes the negative tension D3-brane and provides an exact matching of ten- and five-dimensional tensions, thus eliminating the objection of Ref. [3] towards supersymmetry of the overall scenario. As a result, it is now understood that the brane-world is in fact supersymmetric, and has a natural D3-brane realization in the context of ten-dimensional type IIB theory.

Because of the importance of the breathing mode scalar to supersymmetric realizations of the brane-world, such massive supergravity models are worth further investigation. In fact, the discussion of [15] assumes a standard superpotential and supersymmetry transformations for the breathing mode. While this is clearly valid on general $D = 5$, $N = 2$ grounds, it is nevertheless enlightening to obtain the supersymmetry of the breathing mode directly from the reduction itself. Ref. [14] has provided the general basis for sphere reductions of supergravities with a breathing mode (and possibly an additional squashing mode as well).

In this paper, we extend the results of [14] by reducing the fermionic supersymmetry transformations on the sphere to obtain the corresponding lower-dimensional supersymmetries. These transformations are necessary for the construction of Killing spinors in the presence of the breathing mode, and hence play a key role in understanding the supersymmetry of the brane-world.

The squashing modes considered in [14] correspond to distorting along a $U(1)$ fiber in the cases where the odd-dimensional spheres S^{2n+1} may be written as a $U(1)$ bundle over CP^n . This also may be seen as breaking the $SO(2n+2)$ isometry group (which is also the R -symmetry group) down to $SU(n+1) \times U(1)$. For both the S^5 compactification of type IIB theory and the S^7 compactification of $D=11$ supergravity, this squashing corresponds to the breaking of $N=8$ to $N=2$ supersymmetry through the retention of $SU(n+1)$ singlets only. What makes the fermion supersymmetries non-trivial in these $U(1)$ fibration cases is the fact that the $N=2$ supercharges are charged under this $U(1)$. Since this charge corresponds to Kaluza-Klein momentum in the fiber direction, most previous investigations of such fibered spaces have further truncated to the $U(1)$ neutral sector [16, 17, 18] at the expense of losing some or all supersymmetry. For example, in the S^7 case, the neutral Killing spinors give rise to either $N=6$ or $N=0$ supersymmetry, and not the maximal $N=8$ of the complete (untruncated) S^7 compactification [16, 17]. The situation is even more drastic for the S^5 compactification, as there are no fermions at all in the uncharged sector since CP^2 does not admit a spin structure [19, 20, 18].

In this paper, we will take a closer look at the supersymmetry and construction of Killing spinors for such Hopf fibered spheres. By allowing momentum in the fiber direction, and hence $U(1)$ charge, we are able to demonstrate explicitly the full $N=8$ supersymmetry of the S^5 and S^7 compactifications when written in terms of $U(1)$ bundles over CP^n . Keeping $U(1)$ charged spinors, we then truncate to appropriate $N=2$ limits by restricting to only singlets on CP^n . We conjecture that such a truncation to the $N=2$ breathing/squashing multiplet coupled to $N=2$ supergravity could in fact be a consistent truncation of the full Kaluza-Klein spectrum of the sphere compactification.

Some general features of scalars coupled to $N=2$ supergravity may be obtained by considering a bosonic Lagrangian of the form

$$e^{-1}\mathcal{L}_D = R - \frac{1}{2}|\partial\vec{\phi}|^2 - V(\vec{\phi}), \quad (1.1)$$

with expected supersymmetry transformations

$$\begin{aligned} \delta\psi_\mu &= \left[\nabla_\mu - \frac{1}{2(D-2)\sqrt{2}}W\gamma_\mu \right] \varepsilon, \\ \delta\vec{\lambda} &= \frac{1}{2} \left[\gamma \cdot \partial\vec{\phi} + \sqrt{2}\partial_{\vec{\phi}}W \right] \varepsilon. \end{aligned} \quad (1.2)$$

The potential V may be obtained from the superpotential through the relation

$$V = \left| \partial_{\vec{\phi}}W \right|^2 - \frac{(D-1)}{2(D-2)}W^2. \quad (1.3)$$

This is an example of a Ward-like identity for the scalars in a supergravity theory [21]. Of course, this $N=2$ framework is somewhat heuristic without a complete specification of the supermultiplets (*e.g.* vector, tensor or hyper), fields and transformations. Nevertheless, we will verify that the $N=2$ truncations derived below satisfy the above relations.

In the following section, we examine the S^5 reduction of ten-dimensional type IIB supergravity, both with and without the inclusion of a squashing mode. For the squashing mode,

generated by fibering $U(1)$ over CP^2 , we discuss generalized spin structures and construct appropriately charged Killing spinors. Then in sections 3 and 4 we consider the S^7 and S^4 compactifications of eleven dimensional supergravity. For the former we also include a squashing mode, while for the latter this is not possible. Finally, we conclude with some discussion on the implications for supersymmetry of the brane-world scenario.

2 S^5 Reduction of $D = 10$ type IIB supergravity

Since the five-dimensional case is of immediate attention for the brane-world, we first consider the S^5 compactification of type IIB supergravity. Even with inclusion of the breathing mode, the sphere reduction proceeds via a Freund-Rubin ansatz for the self-dual 5-form. With the 5-form field-strength active, the type IIB supergravity does not admit a covariant Lagrangian formulation. Thus we instead work with the equations of motion. With only the metric and 5-form active, the relevant bosonic equations of motion are:

$$\begin{aligned}\hat{R}_{MN} &= \frac{1}{2} \frac{1}{2 \cdot 4!} F_{MPQRS} F_N{}^{PQRS}, \\ dF_{[5]} &= 0, \\ F_{[5]} &= *F_{[5]}.\end{aligned}\tag{2.1}$$

In addition, the supersymmetry variation of the gravitino is given (on this background) by:

$$\delta\hat{\psi}_M = \left[\hat{\nabla}_M + \frac{i}{16 \cdot 5!} F_{NPQRS} \Gamma^{NPQRS} \Gamma_M \right] \hat{\epsilon},\tag{2.2}$$

where the type IIB spinors are chiral,

$$\Gamma_{11} \hat{\epsilon} = \hat{\epsilon}, \quad \Gamma_{11} \hat{\psi}_M = \hat{\psi}_M.\tag{2.3}$$

The transformation of the ten-dimensional dilatino vanishes trivially for this subset of fields, and hence may be ignored. Note that explicit ten-dimensional quantities are denoted with a caret to avoid confusion with their five-dimensional counterparts.

2.1 The breathing mode reduction

The breathing mode reduction proceeds along the lines of the Kaluza-Klein ansatz of Ref. [14]:

$$\begin{aligned}ds_{10}^2 &= e^{2\alpha\varphi} ds_5^2 + e^{2\beta\varphi} dS^2(S^5), \\ F_{[5]} &= 4me^{8\alpha\varphi} \epsilon_{[5]} + 4m\epsilon_{[5]}(S^5),\end{aligned}\tag{2.4}$$

where

$$\alpha = \frac{1}{4} \sqrt{\frac{5}{3}}, \quad \beta = -\frac{3}{5} \alpha.\tag{2.5}$$

The resulting five-dimensional theory is described by a Lagrangian of the form

$$e^{-1} \mathcal{L}_5 = R - \frac{1}{2} (\partial\varphi)^2 - V(\varphi),\tag{2.6}$$

where $V(\varphi)$ has the double exponential form

$$V = 8m^2 e^{8\alpha\varphi} - R_5 e^{\frac{16}{5}\alpha\varphi}.\tag{2.7}$$

R_5 is the Ricci scalar of S^5 , and may be viewed as a parameter of the compactification. This potential has an AdS_5 minimum at

$$e^{\frac{24}{5}\alpha\varphi_*} = \frac{R_5}{20m^2}. \quad (2.8)$$

We now wish to reduce the $D = 10$ gravitino variation, (2.2), to obtain the corresponding variations in the S^5 reduced theory. To carry out this reduction, we perform a convenient $5 + 5$ decomposition of the ten-dimensional Dirac matrices (in tangent space):

$$\Gamma^{\overline{M}} = (\gamma^{\overline{\mu}} \otimes 1 \otimes \sigma^1, 1 \otimes \tilde{\gamma}^{\overline{a}} \otimes \sigma^2) \quad (2.9)$$

where

$$\begin{aligned} \{\gamma^{\overline{\mu}}, \gamma^{\overline{\nu}}\} &= 2\eta^{\overline{\mu}\overline{\nu}}, \\ \{\tilde{\gamma}^{\overline{a}}, \tilde{\gamma}^{\overline{b}}\} &= 2\delta^{\overline{a}\overline{b}}. \end{aligned} \quad (2.10)$$

Here $\mu, \nu, \dots = 0, 1, 2, 3, 4$ are spacetime indices with $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and $a, b, \dots = 5, 6, 7, 8, 9$ are internal indices with $\tilde{\gamma}^9 \equiv \tilde{\gamma}^5\tilde{\gamma}^6\tilde{\gamma}^7\tilde{\gamma}^8$. The ten-dimensional chirality operator is $\Gamma^{11} = \Gamma^0 \dots \Gamma^9 = 1 \otimes 1 \otimes \sigma^3$, so that type IIB spinors satisfying (2.3) may be written as, *e.g.*

$$\hat{\epsilon} = \epsilon \otimes \eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2.11)$$

For the breathing mode reduction, (2.4), we compute

$$\begin{aligned} F_{MNPQR}\Gamma^{MNPQR} &= 4 \cdot 5! m e^{3\alpha\varphi} [\Gamma^{01234} + \Gamma^{56789}] \\ &= -4i \cdot 5! m e^{3\alpha\varphi} [1 \otimes 1 \otimes (\sigma^1 + i\sigma^2)]. \end{aligned} \quad (2.12)$$

Furthermore, the ten-dimensional covariant derivative decomposes as

$$\begin{aligned} \hat{\nabla}_\mu &= \nabla_\mu \otimes 1 \otimes 1 + \frac{\alpha}{2} \gamma_\mu{}^\nu \partial_\nu \varphi \otimes 1 \otimes 1, \\ \hat{\nabla}_a &= 1 \otimes \tilde{\nabla}_a \otimes 1 + \frac{3i\alpha}{10} e^{-\frac{8}{5}\alpha\varphi} \gamma^\mu \partial_\mu \varphi \otimes \tilde{\gamma}_a \otimes \sigma^3. \end{aligned} \quad (2.13)$$

Hence from (2.2) we obtain the gravitino transformations

$$\begin{aligned} \delta\hat{\psi}_\mu &= \left[\nabla_\mu + \frac{\alpha}{2} \gamma_\mu{}^\nu \partial_\nu \varphi + \frac{m}{4} e^{4\alpha\varphi} \gamma_\mu \otimes 1 \otimes (1 + \sigma^3) \right] \hat{\epsilon}, \\ \delta\hat{\psi}_a &= \left[\tilde{\nabla}_a + \frac{3i\alpha}{10} e^{-\frac{8}{5}\alpha\varphi} \gamma^\mu \partial_\mu \varphi \otimes \tilde{\gamma}_a \otimes \sigma^3 + \frac{im}{4} e^{\frac{12}{5}\alpha\varphi} \tilde{\gamma}_a \otimes (1 + \sigma^3) \right] \hat{\epsilon}. \end{aligned} \quad (2.14)$$

Note that, for simplicity of notation, the tensor product structure in spinor space is hidden in the terms which act only on a single spinor subspace. The chiral structure of the IIB theory is apparent in (2.14), where, in agreement with (2.3), $\hat{\psi}_M$ must have σ^3 eigenvalue $+1$ to transform properly along with a self-dual $F_{[5]}$. Taking this into account, and working with Weyl spinors of the form (2.11), the above simplifies to

$$\begin{aligned} \delta\hat{\psi}_\mu &= \left[\nabla_\mu + \frac{\alpha}{2} \gamma_\mu{}^\nu \partial_\nu \varphi + \frac{m}{2} e^{4\alpha\varphi} \gamma_\mu \right] \hat{\epsilon}, \\ \delta\hat{\psi}_a &= \left[\tilde{\nabla}_a + \frac{3i\alpha}{10} e^{-\frac{8}{5}\alpha\varphi} \gamma^\mu \partial_\mu \varphi \otimes \tilde{\gamma}_a + \frac{im}{2} e^{\frac{12}{5}\alpha\varphi} \tilde{\gamma}_a \right] \hat{\epsilon}. \end{aligned} \quad (2.15)$$

For this Kaluza-Klein reduction, the ten-dimensional spinors $\hat{\psi}_M$ and $\hat{\epsilon}$ may be decomposed in terms of Killing spinors on the sphere. For the n -sphere, such spinors satisfy

$$\left[\tilde{\nabla}_a \pm \frac{i}{2} \tilde{\gamma}_a \sqrt{\frac{R_n}{n(n-1)}} \right] \eta = 0, \quad (2.16)$$

where R_n is the Ricci scalar which fixes the size of the sphere. Note that Killing spinors may be found for either sign of the second term in (2.16), corresponding to the orientation of the sphere. Specializing to the case at hand, η is a complex four-component spinor. Thus there are four independent (complex) Killing spinors on S^5 , giving the expected $D = 5$, $N = 8$ supersymmetry of the round-sphere compactification.

We proceed by using (2.16) to eliminate the covariant derivative on S^5 in (2.15). As a result, we find that $\hat{\psi}_a$ has the property of being a spinorial superpartner to the breathing mode φ , while $\hat{\psi}_\mu$ survives as the five-dimensional gravitino variation. The breathing mode ‘dilatino’ may be normalized by defining

$$\lambda_i^{(5)} \otimes \eta^i = -\frac{i}{3\alpha} e^{\frac{11}{10}\alpha\varphi} \tilde{\gamma}^a \hat{\psi}_a, \quad (2.17)$$

resulting in the transformation

$$\delta \lambda_i^{(5)} = \frac{1}{2} \left[\gamma \cdot \partial \varphi + \frac{5}{3\alpha} \left(m e^{4\alpha\varphi} \mp \sqrt{\frac{R_5}{20}} e^{\frac{8}{5}\alpha\varphi} \right) \right] \varepsilon_i^{(5)}, \quad (2.18)$$

where

$$\varepsilon_i^{(5)} \otimes \eta^i = e^{-\frac{1}{2}\alpha\varphi} \hat{\epsilon}. \quad (2.19)$$

Here, $i = 1, 2, 3, 4$ labels the Killing spinor (where all spinors are taken to be Dirac), indicating the trivial $N = 8$ structure. The scaling of the supersymmetry parameter is natural in a Kaluza-Klein context, and partially eliminates the $\partial_\nu \varphi$ term in the reduction of $\delta \hat{\psi}_\mu$ in (2.15). Similarly, the five-dimensional gravitino takes on the shifted form

$$\psi_{i\mu}^{(5)} \otimes \eta^i = e^{-\frac{1}{2}\alpha\varphi} \hat{\psi}_\mu - \alpha \gamma_\mu \lambda_i^{(5)} \otimes \eta^i, \quad (2.20)$$

so that its supersymmetry transformation has the form

$$\delta \psi_{i\mu}^{(5)} = \left[\nabla_\mu - \frac{1}{6} \left(2m e^{4\alpha\varphi} \mp 5 \sqrt{\frac{R_5}{20}} e^{\frac{8}{5}\alpha\varphi} \right) \gamma_\mu \right] \varepsilon_i^{(5)}. \quad (2.21)$$

Focusing on a particular component, say $i = 1$, we can write the above transformations in the form, (1.2), appropriate to $D = 5$, $N = 2$ supergravity:

$$\begin{aligned} \delta \psi_\mu &= \left[\nabla_\mu - \frac{1}{6\sqrt{2}} W \gamma_\mu \right] \varepsilon, \\ \delta \lambda &= \frac{1}{2} \left[\gamma \cdot \partial \varphi + \sqrt{2} \partial_\varphi W \right] \varepsilon, \end{aligned} \quad (2.22)$$

where one can identify the superpotential as

$$W = \sqrt{2} \left[2m e^{4\alpha\varphi} \mp 5 \sqrt{\frac{R_5}{20}} e^{\frac{8}{5}\alpha\varphi} \right]. \quad (2.23)$$

Furthermore, it is easy to check that this superpotential and the potential (2.7) are related by the five-dimensional version of the identity (1.3):

$$V = (\partial_\varphi W)^2 - \frac{2}{3} W^2. \quad (2.24)$$

Note that the potential itself is unaffected by the sign ambiguity present in (2.23).

2.2 Squashing the five-sphere

Note that S^{2n+1} can be written as a $U(1)$ bundle over CP^n , where in the present case $n = 2$. This suggests the inclusion of a squashing mode, corresponding to a breaking of the $SO(6)$ isometry group of the round S^5 to $SU(3) \times U(1)$. This construction may be made explicit by first reducing the type IIB theory on S^1 , and then reducing further from nine to five dimensions on CP^2 . We follow the procedure given in [18, 14] while paying attention to the fermion supersymmetries.

At this point, it is important to realize that CP^{2n} does not admit a spin structure. Thus, at first sight, the reduction over CP^2 results in a five-dimensional model without any fermions, and in particular without supersymmetry [18]. However we know that this compactification admits fermions and is supersymmetric, because in the appropriate limit it is nothing but reduction on the round S^5 . The resolution of this difficulty is the realization that, while CP^2 does not admit a spin structure, it nevertheless does admit a spin^c structure [19, 20]. This essentially indicates that spinors on CP^2 are charged under the $U(1)$ fiber. As $U(1)$ charge corresponds to momentum along the S^1 direction, such states may be considered ‘massive’ in $D = 9$, and hence are usually truncated out in an ordinary Kaluza-Klein compactification. However, since we are interested in the supersymmetry of this model, we must retain the $U(1)$ charged fermions by allowing for dependence on the circle coordinate. It is worth noting, though, that the bosonic fields may be truncated at the massless Kaluza-Klein level since the background of interest, namely the squashing mode solution of [14], is already complete in the $U(1)$ neutral sector. This simplifies the situation, as otherwise a complete reduction with all massive Kaluza-Klein states would be considerably more involved.

Even with this momentum dependence in mind, the reduction to $D = 9$ proceeds straightforwardly. For a reduction on S^1 , we write

$$ds_{10}^2 = e^{2a\varphi} ds_9^2 + e^{2b\varphi} (dz + \mathcal{A}_M dx^M)^2, \quad (2.25)$$

where now M, N are nine dimensional spacetime indices and m, n are $SO(1, 8)$ tangent space indices. The circle direction is given by z and 9 for curved and tangent space values respectively. The constants a and b are

$$a = -\frac{1}{4\sqrt{7}}, \quad b = -7a. \quad (2.26)$$

With only these fields and $F_{[5]}$ active, the reduction of the IIB equations of motion, (2.1), yield a set of nine-dimensional equations that may be derived from the Lagrangian [14]

$$e^{-1}\mathcal{L}_9 = R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{-16a\varphi}\mathcal{F}_{[2]}^2 - \frac{1}{48}e^{8a\varphi}F_{[4]}^2 - \frac{1}{2}e^{-1} * (F_{[4]} \wedge F_{[4]} \wedge \mathcal{A}_{[1]}). \quad (2.27)$$

Because of its self-dual nature, $F_{[5]}$ reduces to a single four-form field strength, $F_{[4]}$, given by $F_{[4]MNPQ} \equiv F_{[5]MNPQz}$.

For the fermion variations, it is convenient to decompose the ten-dimensional Dirac matrices (in tangent space) as

$$\hat{\Gamma}^{\hat{m}} = (\Gamma^m \otimes \sigma^1, 1 \otimes \sigma^2) \quad (2.28)$$

with $\Gamma^8 = i\Gamma^0 \dots \Gamma^7$. For this choice, $\hat{\Gamma}^{11} = 1 \otimes \sigma^3$, so that IIB spinors have the same form as before. In the following expressions, whenever vielbeins and metric factors are hidden, they are taken to be either ten- or nine-dimensional entities as appropriate. Thus, *e.g.*,

$$\hat{\Gamma}_M = e^{a\varphi}\Gamma_M + e^{b\varphi}\mathcal{A}_M\Gamma^9, \quad \hat{\Gamma}_z = e^{b\varphi}\Gamma^9. \quad (2.29)$$

This shifting of quantities by the $U(1)$ field is standard in Kaluza-Klein reductions.

The gravitino variation, (2.2), splits into both spin- $\frac{3}{2}$ and spin- $\frac{1}{2}$ pieces in $D = 9$. Using the identity

$$F_{\hat{M}\hat{N}\hat{P}\hat{Q}\hat{R}}\hat{\Gamma}^{\hat{M}\hat{N}\hat{P}\hat{Q}\hat{R}} = 5e^{3a\varphi}F_{[4]} \cdot \Gamma\Gamma^9(1 - \hat{\Gamma}^{11}), \quad (2.30)$$

and accounting for the chirality of $\hat{\epsilon}$, (2.3), we obtain after some manipulation

$$\begin{aligned} \delta\psi_M^{(9)} &= \left[D_M + \frac{i}{7}e^{8a\varphi}\Gamma_M\partial_z - \frac{i}{56}e^{-8a\varphi}(\Gamma_M{}^{NP} - 12\delta_M^N\Gamma^P)\mathcal{F}_{NP} \right. \\ &\quad \left. - \frac{1}{56 \cdot 4!}e^{4a\varphi}(\Gamma_M\Gamma^{NPQR} - 7\Gamma^{NPQR}\Gamma_M)F_{NPQR} \right] \epsilon^{(9)}, \\ \delta\lambda^{(9)} &= \frac{1}{2} \left[\Gamma \cdot \partial\varphi - \frac{2i}{7a}e^{8a\varphi}\partial_z + \frac{i}{28a}e^{-8a\varphi}\mathcal{F}_{[2]} \cdot \Gamma + \frac{1}{28 \cdot 4!a}e^{4a\varphi}F_{[4]} \cdot \Gamma \right] \epsilon^{(9)}, \end{aligned} \quad (2.31)$$

where the shifted quantities are defined as

$$\begin{aligned} \psi_M^{(9)} &= e^{-\frac{1}{2}a\varphi}(\hat{\psi}_M - \mathcal{A}_M\hat{\psi}_z) + ia\Gamma_M\Gamma^9\lambda^{(9)}, \\ \lambda^{(9)} &= -\frac{i}{7a}e^{\frac{15}{2}a\varphi}\hat{\psi}_z, \\ \epsilon^{(9)} &= e^{-\frac{1}{2}a\varphi}\hat{\epsilon}. \end{aligned} \quad (2.32)$$

Here, $D_M = \nabla_M - \mathcal{A}_M\partial_z$ is the $U(1)$ covariant derivative for charged spinors. This modification, as well as the inclusion of terms proportional to ∂_z in the above, fully accounts for the possible momentum dependence in the z direction. Other than for this z dependence, after some rearrangement, these expressions agree with the transformations derived in [22].

With the above transformations out of the way, we now proceed to five dimensions using the ansatz [14]

$$\begin{aligned} ds_9^2 &= e^{2\alpha f}ds_5^2 + e^{2\beta f}ds^2(CP^2), \\ F_{[4]} &= 4m\epsilon_{[4]}(CP^2), \\ \mathcal{F}_{[2]} &= 2\mu J_{[2]}(CP^2), \end{aligned} \quad (2.33)$$

where

$$\alpha = \sqrt{\frac{2}{21}}, \quad \beta = -\frac{3}{4}\alpha, \quad (2.34)$$

and $J_{[2]}$ is the Kahler form satisfying $J_{ac}J^c_b = -g_{ab}$ with g_{ab} the standard metric on CP^2 . The reduced bosonic Lagrangian has the form

$$e^{-1}\mathcal{L}_5 = R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial f)^2 - V(\varphi, f), \quad (2.35)$$

with

$$V(\varphi, f) = 8m^2e^{8a\varphi+8\alpha f} + 4\mu^2e^{-16a\varphi+5\alpha f} - R_4e^{\frac{7}{2}\alpha f}. \quad (2.36)$$

Note that we have corrected a factor of four in the second term of V .

The $D = 9$ Dirac matrices may now be split into space-time and CP^2 components

$$\Gamma^M = (\gamma^\mu \otimes \tilde{\gamma}^5, 1 \otimes \tilde{\gamma}^a), \quad (2.37)$$

where the chirality operator on CP^2 is $\tilde{\gamma}^5 = \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^4$. Then, as in the round S^5 compactification, we obtain the decomposition

$$\begin{aligned} D_\mu^{(9)} &= \nabla_\mu \otimes 1 + \frac{\alpha}{2} \gamma_\mu^\nu \partial_\nu f \otimes 1, \\ D_a^{(9)} &= 1 \otimes \tilde{D}_a - \frac{3\alpha}{8} e^{-\frac{7}{4}\alpha f} \gamma \cdot \partial f \otimes \tilde{\gamma}_a \tilde{\gamma}^5. \end{aligned} \quad (2.38)$$

We also make use of the identities

$$F_{[4]} \cdot \Gamma = 4 \cdot 4! m e^{3\alpha f} \tilde{\gamma}^5 \quad (2.39)$$

and

$$\mathcal{F}_{[2]} \cdot \Gamma = -2i\mu e^{\frac{3}{2}\alpha f} \tilde{Q} \tilde{\gamma}^5, \quad (2.40)$$

where

$$\tilde{Q} \equiv iJ_{[2]} \cdot \tilde{\gamma} \tilde{\gamma}^5. \quad (2.41)$$

The relation between $U(1)$ charge and momentum in the z direction can be made more precise. Following [17], we note that the period of z must satisfy

$$\Delta z = \int \mathcal{F}_{[2]} = 2\mu \int J. \quad (2.42)$$

Since R_4 is defined as the Ricci scalar of CP^2 , we have $R_{ab} = \frac{R_4}{4} g_{ab}$, so that $\rho = \frac{R_4}{4} J$ where ρ is the Ricci form. Using $c_1 = \frac{1}{2\pi} \int \rho$ and $c_1(CP^n) = n + 1$ where c_1 is the first Chern class, we finally obtain

$$\Delta z = \frac{48\pi\mu}{R_4}. \quad (2.43)$$

Thus, defining the circle radius L by $z = z + 2\pi L$, we have $L = 24\mu/R_4$. As a result, for a mode expansion in harmonics of the form $e^{iqz/L}$, we may replace ∂_z by

$$\partial_z = \frac{iq}{L} = i \left(\frac{R_4}{24\mu} \right) q \quad (2.44)$$

where q may be considered to be the $U(1)$ charge.

Putting everything together, we find

$$\begin{aligned} \delta\psi_\mu^{(9)} &= \left[\nabla_\mu + \frac{\alpha}{2} \gamma_\mu^\nu \partial_\nu f - \frac{1}{7} \left(\frac{R_4}{24\mu} \right) q e^{8a\varphi + \alpha f} \gamma_\mu \otimes \tilde{\gamma}^5 \right. \\ &\quad \left. - \frac{\mu}{28} e^{-8a\varphi + \frac{5}{2}\alpha f} \gamma_\mu \otimes \tilde{Q} + \frac{3m}{7} e^{4a\varphi + 4\alpha f} \gamma_\mu \right] \epsilon^{(9)}, \\ \delta\psi_a^{(9)} &= \left[\tilde{D}_a - \frac{3\alpha}{8} e^{-\frac{7}{4}\alpha f} \gamma \cdot \partial f \otimes \tilde{\gamma}_a \tilde{\gamma}^5 - \frac{1}{7} \left(\frac{R_4}{24\mu} \right) q e^{8a\varphi - \frac{3}{4}\alpha f} \tilde{\gamma}_a \right. \\ &\quad \left. - \frac{i\mu}{28} e^{-8a\varphi + \frac{3}{4}\alpha f} (\tilde{\gamma}_a^{bc} - 12\delta_a^b \tilde{\gamma}^c) J_{bc} - \frac{4m}{7} e^{4a\varphi + \frac{9}{4}\alpha f} \tilde{\gamma}_a \tilde{\gamma}^5 \right] \epsilon^{(9)}, \\ \delta\lambda^{(9)} &= \frac{1}{2} e^{-\alpha f} \tilde{\gamma}^5 \left[\gamma \cdot \partial \varphi + \frac{2}{7a} \left(\frac{R_4}{24\mu} \right) q e^{8a\varphi + \alpha f} \tilde{\gamma}^5 \right. \\ &\quad \left. + \frac{\mu}{14a} e^{-8a\varphi + \frac{5}{2}\alpha f} \tilde{Q} + \frac{m}{7a} e^{4a\varphi + 4\alpha f} \right] \epsilon^{(9)}. \end{aligned} \quad (2.45)$$

To proceed, we now need to determine the form of the Killing spinors on the Hopf fibration of S^5 . This may be done by realizing that the conventional Freund-Rubin compactification on the round S^5 is obtained when the breathing and squashing modes are turned off. This occurs when both φ and f are sitting at the $N = 8$ critical point of the potential (2.36), corresponding to the constant values

$$\mu^2 = m^2 e^{24a\varphi_* + 3\alpha f_*}, \quad \left(\frac{R_4}{24}\right) = m^2 e^{8a\varphi_* + \frac{9}{2}\alpha f_*}. \quad (2.46)$$

For this case, the gradient terms drop out from (2.45), and one finds

$$\begin{aligned} \delta\lambda^{(9)} &= \frac{1}{28a} e^{\frac{3}{4}\alpha f_*} \tilde{\gamma}^5 \sqrt{\frac{R_4}{24}} (4q\tilde{\gamma}^5 + \tilde{Q} \pm 2) \epsilon^{(9)}, \\ \delta(\psi_a^{(9)} + a e^{-\frac{3}{4}\alpha f_*} \tilde{\gamma}_a \lambda^{(9)}) &= \left[\tilde{D}_a - \frac{1}{2} \sqrt{\frac{R_4}{24}} (\pm \tilde{\gamma}_a \tilde{\gamma}^5 - i J_{ab} \tilde{\gamma}^b) \right] \epsilon^{(9)}. \end{aligned} \quad (2.47)$$

The sign ambiguity arises by considering the two cases,

$$\mu = \pm m e^{12\alpha\varphi_* + \frac{3}{2}\alpha f_*}, \quad (2.48)$$

corresponding to the choice of orientation of S^5 . Now, the vanishing of $\delta\lambda^{(9)}$ then imposes the condition,

$$(4q\tilde{\gamma}^5 + \tilde{Q} \pm 2)\eta = 0, \quad (2.49)$$

on Killing spinors η . From the definition of \tilde{Q} , (2.41), it is easy to verify that it has eigenvalues $(-4, 0, 0, 4)$, with corresponding $\tilde{\gamma}^5$ eigenvalues $(-1, 1, 1, -1)$. Focusing on the positive sign in (2.49), we see that it may be satisfied for $U(1)$ charges $q = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{2})$. Note that this verifies the decomposition of the complex spinor representation $4 \rightarrow 3_{-1/2} + 1_{3/2}$ under the split $SO(6) \supset SU(3) \times U(1)$. For the negative sign, we would obtain instead $q = (-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which is the charge conjugate of the above.

It remains to consider the gravitino variation on CP^2 . Here it is easier not to consider the variation (2.47) directly, but rather to check the integrability condition

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = \frac{1}{4} R_{abcd} \tilde{\gamma}^{cd} + \frac{R_4}{48} [-\tilde{\gamma}_{ab} - 2i J_{ab} (2q \pm \tilde{\gamma}^5) - J_{ac} J_{bd} \tilde{\gamma}^{cd}]. \quad (2.50)$$

Substituting the Riemann tensor on CP^2 ,

$$R_{abcd} = \frac{R_4}{24} [g_{ac} g_{bd} - g_{ad} g_{bc} + J_{ac} J_{bd} - J_{ad} J_{bc} + 2 J_{ab} J_{cd}], \quad (2.51)$$

we then find

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = -i \frac{R_4}{48} J_{ab} \tilde{\gamma}^5 (4q\tilde{\gamma}^5 + \tilde{Q} \pm 2), \quad (2.52)$$

which indeed vanishes for Killing spinors satisfying (2.49). We have thus found the expected four independent complex Killing spinors for the $U(1)$ fibered CP^2 construction.

Following the procedure applied previously to the round S^5 , we now return to the full supersymmetry transformations, (2.45), and decompose the various fermions in terms of the above set of Killing spinors. Defining

$$\begin{aligned} \varepsilon^{(5)} \otimes \eta &= e^{-\frac{1}{2}\alpha f} \epsilon^{(9)}, \\ \lambda^{(5)} \otimes \eta &= e^{\frac{1}{2}\alpha f} \tilde{\gamma}^5 \lambda^{(9)}, \\ \chi^{(5)} \otimes \eta &= \frac{1}{3\alpha} e^{\frac{5}{4}\alpha f} \tilde{\gamma}^a \tilde{\gamma}^5 \psi_a^{(9)}, \\ \psi_\mu^{(5)} \otimes \eta &= e^{-\frac{1}{2}\alpha f} \psi_\mu^{(9)} - \alpha \gamma_\mu \chi^{(5)} \otimes \eta, \end{aligned} \quad (2.53)$$

and using (2.49) to eliminate the charge q , we find the resulting set of five-dimensional supersymmetry transformations:

$$\begin{aligned}
\delta\lambda^{(5)} &= \frac{1}{2}\left[\gamma \cdot \partial\varphi - \frac{1}{14a}\left(\frac{R_4}{24\mu}\right)(\tilde{Q} \pm 2)e^{8a\varphi+\alpha f} + \frac{\mu}{14a}\tilde{Q}e^{-8a\varphi+\frac{5}{2}\alpha f} + \frac{m}{7a}e^{4a\varphi+4\alpha f}\right]\varepsilon^{(5)}, \\
\delta\chi^{(5)} &= \frac{1}{2}\left[\gamma \cdot \partial f + \frac{1}{3\alpha}\sqrt{\frac{R_4}{24}}(\tilde{Q} \mp 4)e^{\frac{7}{4}\alpha f} - \frac{2}{21\alpha}\left(\frac{R_4}{24\mu}\right)(\tilde{Q} \pm 2)e^{8a\varphi+\alpha f} \right. \\
&\quad \left. - \frac{5\mu}{21\alpha}\tilde{Q}e^{-8a\varphi+\frac{5}{2}\alpha f} + \frac{32m}{21\alpha}e^{4a\varphi+4\alpha f}\right]\varepsilon^{(5)}, \\
\delta\psi_\mu^{(5)} &= \left[\nabla_\mu - \frac{1}{6}\sqrt{\frac{R_4}{24}}(\tilde{Q} \mp 4)e^{\frac{7}{4}\alpha f}\gamma_\mu + \frac{1}{12}\left(\frac{R_4}{24\mu}\right)(\tilde{Q} \pm 2)e^{8a\varphi+\alpha f}\gamma_\mu \right. \\
&\quad \left. + \frac{\mu}{12}\tilde{Q}e^{-8a\varphi+\frac{5}{2}\alpha f}\gamma_\mu - \frac{m}{3}e^{4a\varphi+4\alpha f}\gamma_\mu\right]\varepsilon^{(5)}. \tag{2.54}
\end{aligned}$$

(one may consider there to be four sets of such equations—one for each of the four eigenvalues of \tilde{Q} .)

Remarkably, these transformations follow the $N = 2$ form, (1.2), with ‘superpotential’

$$W = \sqrt{2}\left[\sqrt{\frac{R_4}{24}}(\tilde{Q} \mp 4)e^{\frac{7}{4}\alpha f} - \frac{1}{2}\left(\frac{R_4}{24\mu}\right)(\tilde{Q} \pm 2)e^{8a\varphi+\alpha f} - \frac{\mu}{2}\tilde{Q}e^{-8a\varphi+\frac{5}{2}\alpha f} + 2me^{4a\varphi+4\alpha f}\right]. \tag{2.55}$$

While this holds for all valid choices of \tilde{Q} eigenvalue, based on the truncation to the $N = 2$ model with squashing mode, we are only interested in retaining the $SU(3)$ singlet state $1_{3/2}$, with $\tilde{Q} = 4$ (or $1_{-3/2}$ with $\tilde{Q} = -4$). In this case, the superpotential reads

$$W = \sqrt{2}\left[2me^{4a\varphi+4\alpha f} \mp 2\mu e^{-8a\varphi+\frac{5}{2}\alpha f} \mp \frac{R_4}{8\mu}e^{8a\varphi+\alpha f}\right], \tag{2.56}$$

and it may be verified to satisfy the identity (1.3). This actual reduction of the type IIB supersymmetry to $D = 5$, $N = 2$ verifies the form of the superpotential assumed in [15].

Note that, at this stage, the scalars φ and f are still linear combinations of the $E_0 = 8$ breathing and $E_0 = 6$ squashing mode on S^5 . An $O(2)$ rotation, given in [14], may be used to disentangle these two modes. From the $N = 2$ point of view, the breathing and squashing modes (φ, f) , along with the fermions $(\lambda^{(5)}, \chi^{(5)})$, belong to a massive vector representation of $SU(2, 2|1)^1$. Denoting AdS_5 representations by $D(E_0, j_1, j_2; r)$ where r is the $U(1)$ charge, the content of this vector multiplet is given by

$$\begin{aligned}
\mathcal{D}(6, 0, 0; 0) &= D(7, \tfrac{1}{2}, \tfrac{1}{2}; 0) \\
&\quad + D(6\tfrac{1}{2}, \tfrac{1}{2}, 0; -1) + D(6\tfrac{1}{2}, 0, \tfrac{1}{2}; 1) + D(7\tfrac{1}{2}, 0, \tfrac{1}{2}; -1) + D(7\tfrac{1}{2}, \tfrac{1}{2}, 0; 1) \\
&\quad + D(6, 0, 0; 0) + D(7, 0, 0; -2) + D(7, 0, 0; 2) + D(8, 0, 0; 0). \tag{2.57}
\end{aligned}$$

The breathing mode and squashing mode scalars can be identified with $D(8, 0, 0; 0)$ and $D(6, 0, 0; 0)$, respectively. In addition to the fields we have considered, this indicates the presence of a charged scalar, $D(7, 0, 0; -2) + D(7, 0, 0; 2)$, and vector, $D(7, \frac{1}{2}, \frac{1}{2}; 0)$. The latter presumably has its origin in $\mathcal{A}_{[1]}$.

¹Unitary highest weight representations of $SU(2, 2|1)$ were constructed in [23, 24], while those of $SU(2, 2|N/2)$ were investigated in [25, 26, 27, 28]. See also [29, 30].

The Kaluza-Klein spectrum for the round S^5 compactification of type IIB theory was obtained in [31, 32], and falls into unitary representations of $SU(2, 2|4)$. Following the above procedure of squashing the five-sphere, this $N = 8$ supersymmetry may be broken by decomposing $SU(2, 2|4) \supset SU(2, 2|1) \times SU(3) \times U(1)$ and truncating to the $SU(3)$ singlet sector [33]. For the massless supergravity sector, this decomposition yields the $N = 2$ gravity multiplet coupled to a LH+RH chiral multiplet (which contains the type IIB dilaton and Ramond-Ramond scalar):

$$\begin{aligned}\mathcal{D}(3, \tfrac{1}{2}, \tfrac{1}{2}; 0) &= D(4, 1, 1; 0) + D(3\tfrac{1}{2}, 1, \tfrac{1}{2}; -1) + D(3\tfrac{1}{2}, \tfrac{1}{2}, 1; 1) + D(3, \tfrac{1}{2}, \tfrac{1}{2}; 0), \\ \mathcal{D}(3, 0, 0, 2) &= D(3\tfrac{1}{2}, \tfrac{1}{2}, 0; 1) + D(3, 0, 0; 2) + D(4, 0, 0; 0), \\ \mathcal{D}(3, 0, 0, -2) &= D(3\tfrac{1}{2}, 0, \tfrac{1}{2}; -1) + D(3, 0, 0; -2) + D(4, 0, 0; 0).\end{aligned}\tag{2.58}$$

At the first Kaluza-Klein level, the $N = 2$ truncation yields a semi-long LH+RH massive gravitino multiplet:

$$\begin{aligned}\mathcal{D}(4\tfrac{1}{2}, 0, \tfrac{1}{2}, 1) &= D(5\tfrac{1}{2}, \tfrac{1}{2}, 1; 1) + D(5, \tfrac{1}{2}, \tfrac{1}{2}; 0) + D(5, 0, 1; 2) + D(6, 0, 1, 0) \\ &\quad + D(4\tfrac{1}{2}, 0, \tfrac{1}{2}, 1) + D(5\tfrac{1}{2}, 0, \tfrac{1}{2}, -1), \\ \mathcal{D}(4\tfrac{1}{2}, \tfrac{1}{2}, 0, -1) &= D(5\tfrac{1}{2}, 1, \tfrac{1}{2}; -1) + D(5, \tfrac{1}{2}, \tfrac{1}{2}; 0) + D(5, 1, 0; -2) + D(6, 1, 0, 0) \\ &\quad + D(4\tfrac{1}{2}, \tfrac{1}{2}, 0, -1) + D(5\tfrac{1}{2}, \tfrac{1}{2}, 0, 1).\end{aligned}\tag{2.59}$$

Finally, at the second Kaluza-Klein level, the truncation yields precisely the $N = 2$ breathing/squashing multiplet given in (2.57). These decompositions agree with the bosonic sector of the five-dimensional Lagrangian obtained in [14].

3 Reduction of $D = 11$ Supergravity to four dimensions

Our second example of the supersymmetry of breathing mode compactifications concerns the reduction of eleven dimensional supergravity on S^7 [34]. In this case, we start with the bosonic fields, \hat{G}_{MN} and $\hat{F}_{[4]} = dA_{[3]}$, with Lagrangian

$$e^{-1}\mathcal{L}_{11} = \hat{R} - \frac{1}{2 \cdot 4!} F_{[4]}^2 - \frac{1}{6} e^{-1} * (F_{[4]} \wedge F_{[4]} \wedge A_{[3]}).\tag{3.1}$$

For a bosonic background, the $D = 11$ gravitino supersymmetry transformation is given by [35]

$$\delta\hat{\psi}_M = \left[\hat{\nabla}_M - \frac{1}{288} (\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS}) F_{PQRS} \right] \hat{\epsilon}.\tag{3.2}$$

As in the type IIB scenario, we first consider the case of a round S^7 , followed by the turning on of a squashing mode introduced by writing S^7 as a $U(1)$ bundle over CP^3 .

3.1 The breathing mode

Following the general sphere reduction of [14], for the round S^7 we choose the standard ansatz

$$\begin{aligned}ds_{11}^2 &= e^{2\alpha\varphi} ds_4^2 + e^{2\beta\varphi} ds^2(S^7) \\ F_{[4]} &= ce^{6\alpha\varphi} \epsilon_{[4]},\end{aligned}\tag{3.3}$$

where $\epsilon_{[4]}$ is the volume form in the $D = 4$ spacetime. For the case at hand, α and β take on the values

$$\alpha = \frac{\sqrt{7}}{6}, \quad \beta = -\frac{2}{7}\alpha. \quad (3.4)$$

The resulting four-dimensional bosonic Lagrangian reads [14]

$$e^{-1}\mathcal{L}_4 = R - \frac{1}{2}(\partial\varphi)^2 - V(\phi), \quad (3.5)$$

where

$$V = \frac{1}{2}c^2 e^{6\alpha\varphi} - R_7 e^{\frac{18}{7}\alpha\varphi}. \quad (3.6)$$

This potential has an AdS_4 minimum at

$$e^{\frac{24}{7}\alpha\varphi_*} = \frac{6R_7}{7c^2}, \quad (3.7)$$

where R_7 is the Ricci scalar of S^7 .

The reduction of (3.2) is now straightforward, and follows the procedure developed in the previous section. Thus we omit the details, and only point out some salient features of the reduction. For spinors, we start with a natural $4 + 7$ split of the Dirac matrices:

$$\Gamma^M = (\gamma^\mu \otimes 1, \gamma^5 \otimes \tilde{\gamma}^a), \quad (3.8)$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the spacetime chirality matrix (and squares to $+1$). Using this decomposition, the spacetime and sphere components of (3.2) become

$$\begin{aligned} \delta\hat{\psi}_\mu &= \left[\nabla_\mu + \frac{\alpha}{2}\gamma_\mu{}^\nu \partial_\nu \varphi - \frac{i}{6}ce^{3\alpha\varphi}\gamma_\mu\gamma^5 \right] \hat{\epsilon}, \\ \delta\hat{\psi}_a &= \left[\tilde{\nabla}_a - \frac{\alpha}{7}e^{-\frac{9}{7}\alpha\varphi}\gamma^5(\gamma \cdot \partial\varphi) \otimes \tilde{\gamma}_a + \frac{i}{12}ce^{\frac{12}{7}\alpha\varphi}\tilde{\gamma}_a \right] \hat{\epsilon}. \end{aligned} \quad (3.9)$$

Defining the shifted four-dimensional quantities as

$$\begin{aligned} \varepsilon_i^{(4)} \otimes \eta^i &= e^{-\frac{1}{2}\alpha\varphi} \hat{\epsilon}, \\ \lambda_i^{(4)} \otimes \eta^i &= -\frac{1}{2\alpha}e^{\frac{11}{14}\alpha\varphi}\gamma^5\tilde{\gamma}^a\hat{\psi}_a, \\ \psi_{i\mu}^{(4)} \otimes \eta^i &= e^{-\frac{1}{2}\alpha\varphi}\hat{\psi}_\mu - \alpha\gamma_\mu\lambda_i^{(4)} \otimes \eta^i, \end{aligned} \quad (3.10)$$

and making use of the Killing spinor equation on the sphere, (2.16), we finally obtain the four-dimensional supersymmetry variations

$$\begin{aligned} \delta\psi_{i\mu}^{(4)} &= \left[\nabla_\mu + \frac{i}{8} \left(ce^{3\alpha\varphi} \mp 14\sqrt{\frac{R_4}{42}}e^{\frac{9}{7}\alpha\varphi} \right) \gamma_\mu\gamma^5 \right] \varepsilon_i^{(4)}, \\ \delta\lambda_i^{(4)} &= \frac{1}{2} \left[\gamma \cdot \partial\varphi - \frac{7i}{12\alpha} \left(ce^{3\alpha\varphi} \mp 6\sqrt{\frac{R_4}{42}}e^{\frac{9}{7}\alpha\varphi} \right) \gamma^5 \right] \varepsilon_i^{(4)}. \end{aligned} \quad (3.11)$$

The index i runs from 1 to 8, and labels the $D = 4$, $N = 8$ supersymmetry arising from having eight Killing spinors on S^7 . Furthermore, as before, the choice of sign depends on the orientation of the sphere.

The factor $i\gamma^5$ may be rewritten for Majorana spinors in $D = 4$, as appropriate. Regardless, there is a straightforward truncation to $N = 2$ obtained by choosing a single Majorana spinor pair. The resulting transformations, written in a $D = 4$, $N = 2$ language, are

$$\begin{aligned}\delta\psi_\mu &= \left[\nabla_\mu - \frac{1}{4\sqrt{2}}W\gamma_\mu(i\gamma^5)\right]\varepsilon, \\ \delta\lambda &= \frac{1}{2}\left[\gamma\cdot\partial\varphi + \sqrt{2}\partial_\varphi W(i\gamma^5)\right]\varepsilon,\end{aligned}\tag{3.12}$$

where

$$W = -\frac{1}{\sqrt{2}}\left[ce^{3\alpha\varphi} \mp 14\sqrt{\frac{R_7}{42}}e^{\frac{9}{7}\alpha\varphi}\right].\tag{3.13}$$

This superpotential and the potential, (3.6), satisfy the relation (1.3).

3.2 Introduction of a squashing mode

Again since S^7 can be viewed as a $U(1)$ bundle over CP^3 , one can perform the reduction in two steps. We first reduce from eleven dimensions on a circle giving the type IIA theory in ten dimensions. Following this, we may proceed from ten down to four dimensions on CP^3 . This approach to squashing S^7 has been extensively studied in [16, 17]. However note that, as in the S^5 case, we wish to retain states charged under the $U(1)$ fiber. While such states may be regarded as non-perturbative in a type IIA point of view [17], they naturally arise from eleven dimensions and complete the supersymmetry of the compactification.

As seen in [16, 17], the $U(1)$ neutral sector has either $N = 6$ or $N = 0$ supersymmetry in four dimensions, depending on the orientation of S^7 . This is easily seen by considering the decomposition of the spinor 8_s under $SO(8) \supset SU(4) \times U(1)$: $8_s \rightarrow 6_0 + 1_2 + 1_{-2}$ for left squashing and $8_s \rightarrow 4_1 + \bar{4}_{-1}$ for right squashing. Thus in the round S^7 limit, complete $N = 8$ supersymmetry requires the introduction of $U(1)$ charged spinors. Furthermore, we are mainly interested in the truncation of the breathing/squashing system to $N = 2$, which corresponds to the $SU(4)$ singlet sector under the above decomposition. We see that the left squashed compactification has precisely the expected $SU(4)$ singlet supercharges corresponding to $U(1)$ gauged $D = 4$, $N = 2$ supergravity. Curiously, this $N = 2$ supersymmetry is in some sense complementary to the $N = 6$ supersymmetry considered previously in the $U(1)$ neutral sector [16].

There has been a long tradition, starting with Refs. [36, 37], of reducing eleven dimensional supergravity on a circle in order to obtain the type IIA theory. The reduction proceeds with the metric ansatz

$$ds_{11}^2 = e^{2a\phi}ds_{10}^2 + e^{2b\phi}(dz + \mathcal{A}_M dx^M)^2,\tag{3.14}$$

where

$$a = -\frac{1}{12}, \quad b = -8a.\tag{3.15}$$

The resulting type IIA supergravity is described by the bosonic Lagrangian

$$e^{-1}\mathcal{L}_{10} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 2!}e^{\frac{3}{2}\phi}\mathcal{F}_{[2]}^2 - \frac{1}{2\cdot 3!}e^{-\phi}F_{[3]}^2 - \frac{1}{2\cdot 4!}e^{\frac{1}{2}\phi}F_{[4]}^2 - \frac{1}{2}*(F_{[4]} \wedge F_{[4]} \wedge A_{[2]}),\tag{3.16}$$

where $F_{[4]}$ is now shifted, $F_{[4]} = dA_{[3]} - F_{[3]} \wedge \mathcal{A}_{[1]}$.

Eleven-dimensional Dirac matrices may be given in terms of their ten-dimensional counterparts by setting $\Gamma^{10} = \Gamma^{11}$ where $\Gamma^{11} \equiv \Gamma^0 \Gamma^1 \dots \Gamma^9$ is the $D = 10$ chirality operator. In this case, the resulting supersymmetry transformations are [36, 37]

$$\begin{aligned}
\delta\psi_M^{(10)} &= \left[D_M + \frac{1}{8}e^{-\frac{3}{4}\phi}\Gamma_M\Gamma^{11}\partial_z - \frac{1}{64}e^{\frac{3}{4}\phi}(\Gamma_M{}^{NP} - 14\delta_M^N\Gamma^P)\mathcal{F}_{NP}\Gamma^{11} \right. \\
&\quad - \frac{1}{4 \cdot 4!}e^{-\frac{1}{2}\phi}(\Gamma_M{}^{NPQ} - 9\delta_M^N\Gamma^{PQ})F_{NPQ}\Gamma^{11} \\
&\quad \left. - \frac{1}{256}e^{\frac{1}{4}\phi}(\Gamma_M{}^{NPQR} - \frac{20}{3}\delta_M^N\Gamma^{PQR})F_{NPQR} \right] \epsilon^{(10)}, \\
\delta\lambda^{(10)} &= \frac{1}{2} \left[\Gamma \cdot \partial\phi + 3e^{-\frac{3}{4}\phi}\Gamma^{11}\partial_z - \frac{3}{8}e^{\frac{3}{4}\phi}\mathcal{F}_{[2]} \cdot \Gamma\Gamma^{11} \right. \\
&\quad \left. + \frac{1}{12}e^{-\frac{1}{2}\phi}F_3 \cdot \Gamma\Gamma^{11} - \frac{1}{96}e^{\frac{1}{4}\phi}F_4 \cdot \Gamma \right] \epsilon^{(10)}. \tag{3.17}
\end{aligned}$$

The $D = 10$ quantities are related to the original ones by

$$\begin{aligned}
\psi_M^{(10)} &= e^{\frac{1}{24}\phi}(\hat{\psi}_M - \mathcal{A}_M\hat{\psi}_z) + \frac{1}{12}\Gamma_M\lambda^{(10)}, \\
\lambda^{(10)} &= \frac{3}{2}e^{-\frac{17}{24}\phi}\Gamma^{11}\hat{\psi}_z, \\
\epsilon^{(10)} &= e^{\frac{1}{24}\phi}\hat{\epsilon}. \tag{3.18}
\end{aligned}$$

As in the squashing of S^5 , we have retained momentum dependence in the z direction, corresponding to $U(1)$ charged spinors. The $U(1)$ covariant derivative is given by $D_M = \nabla_M - \mathcal{A}_M\partial_z$.

For a reduction on CP^3 , we choose the ansatz [17, 14]

$$\begin{aligned}
ds_{10}^2 &= e^{2\alpha\varphi}ds_4^2 + e^{2\beta\varphi}ds^2(CP^3), \\
F_{[4]} &= ce^{-\frac{1}{2}\phi+6\alpha\varphi}\epsilon_{(4)}, \\
F_{[3]} &= 0, \\
\mathcal{F}_{[2]} &= 2mJ_{[2]}(CP^3), \tag{3.19}
\end{aligned}$$

with

$$\alpha = \frac{\sqrt{3}}{4}, \quad \beta = -\frac{1}{3}\alpha. \tag{3.20}$$

Note that this ansatz sets to zero the $D = 4$ field strengths originating from $F_{[3]}$ and $\mathcal{F}_{[2]}$ (since our interest is only on the breathing/squashing scalars). The resulting $D = 4$ Lagrangian has the form

$$e^{-1}\mathcal{L}_4 = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - V(\phi, \varphi), \tag{3.21}$$

where

$$V = 6m^2e^{\frac{3}{2}\phi+\frac{10}{3}\alpha\varphi} + \frac{1}{2}c^2e^{-\frac{1}{2}\phi+6\alpha\varphi} - R_6e^{\frac{8}{3}\alpha\varphi}. \tag{3.22}$$

The AdS_4 minimum of this potential lies at

$$c^2 = 36m^2e^{2\phi_*-\frac{8}{3}\alpha\varphi_*}, \quad R_6 = 48m^2e^{\frac{3}{2}\phi_*+\frac{2}{3}\alpha\varphi_*}, \tag{3.23}$$

which is in fact the round S^7 vacuum.

For reduction of the supersymmetry variations, (3.17), we may use the decomposition of the Dirac matrices, (3.8), specialized now to the present $10 = 4 + 6$ split. With this split, we find $\Gamma^{11} = -\gamma^5 \otimes \tilde{\gamma}^7$, where $\tilde{\gamma}^7 = i\tilde{\gamma}^1\tilde{\gamma}^2\cdots\tilde{\gamma}^6$. Substituting the ansatz, (3.19), into (3.17), we find

$$\begin{aligned}
\delta\psi_\mu^{(10)} &= \left[\nabla_\mu + \frac{\alpha}{2}\gamma_\mu{}^\nu\partial_\nu\varphi - \frac{i}{32}\gamma_\mu\gamma^5\left(4\left(\frac{R_6}{48m}\right)qe^{-\frac{3}{4}\phi+\alpha\varphi}\tilde{\gamma}^7\right. \right. \\
&\quad \left. \left. + m\tilde{Q}e^{\frac{3}{4}\phi+\frac{5}{3}\alpha\varphi} + 5ce^{-\frac{1}{4}\phi+3\alpha\varphi}\right)\right]\epsilon^{(10)}, \\
\delta\psi_a^{(10)} &= \left[\tilde{D}_a - \frac{\alpha}{6}e^{-\frac{4}{3}\alpha\varphi}\gamma^5\gamma\cdot\partial\varphi\tilde{\gamma}_a - \frac{i}{8}\left(\frac{R_6}{48m}\right)qe^{-\frac{3}{4}\phi-\frac{1}{3}\alpha\varphi}\tilde{\gamma}_a\tilde{\gamma}^7 \right. \\
&\quad \left. + \frac{m}{32}e^{\frac{3}{4}\phi+\frac{1}{3}\alpha\varphi}(\tilde{\gamma}_a{}^{bc} - 14\delta_a^b\tilde{\gamma}^c)J_{bc}\tilde{\gamma}^7 + \frac{3ic}{32}e^{-\frac{1}{4}\phi+\frac{5}{3}\alpha\varphi}\tilde{\gamma}_a\right]\epsilon^{(10)}, \\
\delta\lambda^{(10)} &= \frac{1}{2}e^{-\alpha\varphi}\left[\gamma\cdot\partial\phi - 3i\gamma^5\left(\left(\frac{R_6}{48m}\right)qe^{-\frac{3}{4}\phi+\alpha\varphi}\tilde{\gamma}^7\right. \right. \\
&\quad \left. \left. + \frac{m}{4}\tilde{Q}e^{\frac{3}{4}\phi+\frac{5}{3}\alpha\varphi} - \frac{c}{12}e^{-\frac{1}{4}\phi+3\alpha\varphi}\right)\right]\epsilon^{(10)}. \quad (3.24)
\end{aligned}$$

As in the case of $U(1)$ bundled over CP^2 , we use the definition

$$\tilde{Q} = iJ_{[2]} \cdot \tilde{\gamma}\gamma^7. \quad (3.25)$$

Furthermore, using the same argument as before, we have identified the period along the circle direction, $z = z + 2\pi L$, to be $L = 48m/R_6$ [17]. The $U(1)$ charge is then related to the Kaluza-Klein momentum through $\partial_z = iq/L$.

The Killing spinors η on the Hopf fibered S^7 may be obtained by examination of the variations (3.24) at the $N = 8$ critical point given by (3.23). Corresponding to the two possibilities

$$c = \pm 6me^{\phi_*-\frac{4}{3}\alpha\varphi_*}, \quad (3.26)$$

we find

$$\begin{aligned}
\delta\lambda^{(10)} &= -\frac{3i}{8}e^{\frac{1}{3}\alpha\varphi_*}\gamma^5\sqrt{\frac{R_6}{48}}(4q\tilde{\gamma}^7 + \tilde{Q} \mp 2)\epsilon^{(10)}, \\
\delta(\psi_a^{(10)} - \frac{1}{12}e^{-\frac{1}{3}\alpha\varphi_*}\gamma^5\tilde{\gamma}_a\lambda^{(10)}) &= \left[\tilde{D}_a - \frac{i}{2}\sqrt{\frac{R_6}{48}}(\mp\tilde{\gamma}_a - iJ_{ab}\tilde{\gamma}^b\tilde{\gamma}^7)\right]\epsilon^{(10)}. \quad (3.27)
\end{aligned}$$

As a result, Killing spinors must satisfy

$$(4q\tilde{\gamma}^7 + \tilde{Q} \mp 2)\eta = 0 \quad (3.28)$$

[compare with (2.49)]. For CP^3 , \tilde{Q} has eigenvalues 2 (six times) and -6 (twice). For either the top or bottom choice of sign in (3.28), corresponding to left- or right-squashing respectively, we may find the appropriate $U(1)$ charge q such that the Killing spinor condition is satisfied. The result is shown in Table 3.2, and agrees with the charge assignments obtained from the spinor decompositions $8_s \rightarrow 6_0 + 1_2 + 1_{-2}$ (left-squashing) and $8_s \rightarrow 4_1 + \bar{4}_{-1}$ (right-squashing). Furthermore, it is also straightforward to check the integrability of the gravitino variation on CP^3 , following the procedure given previously for the case of CP^2 .

| \tilde{Q} | $\tilde{\gamma}^7$ | q (left-squashing) | q (right-squashing) |
|-------------|--------------------|----------------------|-----------------------|
| -6 | -1 | -2 | -1 |
| -6 | 1 | 2 | 1 |
| 2 | -1 | 0 | 1 |
| 2 | -1 | 0 | 1 |
| 2 | -1 | 0 | 1 |
| 2 | 1 | 0 | -1 |
| 2 | 1 | 0 | -1 |
| 2 | 1 | 0 | -1 |

Table 1: The \tilde{Q} and $\tilde{\gamma}^7$ eigenvalues and $U(1)$ charges q for eight Killing spinors on the $U(1)$ over CP^3 fibered S^7 .

Using the effective Killing spinor equation,

$$\left(\tilde{\gamma}^a \tilde{D}_a + \frac{i}{2} \sqrt{\frac{R_6}{48}} (\tilde{Q} \pm 6) \right) \eta = 0, \quad (3.29)$$

obtained from (3.28) to eliminate \tilde{D}_a from (3.24), we finally arrive at the set of $D = 4$ supersymmetry variations

$$\begin{aligned} \delta \lambda^{(4)} &= \frac{1}{2} \left[\gamma \cdot \partial \phi - \frac{3i}{4} \gamma^5 \left(- \left(\frac{R_6}{48m} \right) (\tilde{Q} \mp 2) e^{-\frac{3}{4}\phi + \alpha\varphi} + m \tilde{Q} e^{\frac{3}{4}\phi + \frac{5}{3}\alpha\varphi} - \frac{c}{3} e^{-\frac{1}{4} + 3\alpha\varphi} \right) \right] \varepsilon^{(4)}, \\ \delta \chi^{(4)} &= \frac{1}{2} \left[\gamma \cdot \partial \varphi + \frac{i}{16\alpha} \gamma^5 \left(8 \sqrt{\frac{R_6}{48}} (\tilde{Q} \pm 6) e^{\frac{4}{3}\alpha\varphi} - 3 \left(\frac{R_6}{48m} \right) (\tilde{Q} \mp 2) e^{-\frac{3}{4}\phi + \alpha\varphi} \right. \right. \\ &\quad \left. \left. - 5m \tilde{Q} e^{\frac{3}{4}\phi + \frac{5}{3}\alpha\varphi} - 9ce^{-\frac{1}{4} + 3\alpha\varphi} \right) \right] \varepsilon^{(4)}, \\ \delta \psi_\mu^{(4)} &= \left[\nabla_\mu - \frac{i}{8} \gamma_\mu \gamma^5 \left(2 \sqrt{\frac{R_6}{48}} (\tilde{Q} \pm 6) e^{\frac{4}{3}\alpha\varphi} - \left(\frac{R_6}{48m} \right) (\tilde{Q} \mp 2) e^{-\frac{3}{4}\phi + \alpha\varphi} \right. \right. \\ &\quad \left. \left. - m \tilde{Q} e^{\frac{3}{4}\phi + \frac{5}{3}\alpha\varphi} - ce^{-\frac{1}{4} + 3\alpha\varphi} \right) \right] \varepsilon^{(4)}. \end{aligned} \quad (3.30)$$

The four-dimensional spinors are related to the original ten-dimensional ones through

$$\begin{aligned} \varepsilon^{(4)} \otimes \eta &= e^{-\frac{1}{2}\alpha\varphi} \epsilon^{(10)}, \\ \lambda^{(4)} \otimes \eta &= e^{\frac{1}{2}\alpha\varphi} \lambda^{(10)}, \\ \chi^{(4)} \otimes \eta &= -\frac{1}{2\alpha} e^{\frac{5}{6}\alpha\varphi} \gamma^5 \tilde{\gamma}_a \psi_a^{(10)}, \\ \psi_\mu^{(4)} \otimes \eta &= e^{\frac{1}{2}\alpha\varphi} \psi_\mu^{(10)} - \alpha \gamma_\mu \chi^{(4)} \otimes \eta. \end{aligned} \quad (3.31)$$

Once again, these variations may be written in an explicit $D = 4$, $N = 2$ manner, given by (3.12), where the superpotential has the form

$$W = -\frac{1}{\sqrt{2}} \left[-2 \sqrt{\frac{R_6}{48}} (\tilde{Q} \pm 6) e^{\frac{4}{3}\alpha\varphi} + \left(\frac{R_6}{48m} \right) (\tilde{Q} \mp 2) e^{-\frac{3}{4}\phi + \alpha\varphi} + m \tilde{Q} e^{\frac{3}{4}\phi + \frac{5}{3}\alpha\varphi} + ce^{-\frac{1}{4} + 3\alpha\varphi} \right]. \quad (3.32)$$

The only case where supersymmetry survives truncation to the $SU(4)$ singlet sector is for left-squashing (the top sign), in which case the two $\tilde{Q} = -6$ states may be combined into a single $U(1)$ charged Dirac spinor. The resulting $N = 2$ superpotential has the form

$$W = -\frac{1}{\sqrt{2}} \left[ce^{-\frac{1}{4}+3\alpha\varphi} - 6me^{\frac{3}{4}\phi+\frac{5}{3}\alpha\varphi} - \frac{R_6}{6m}e^{-\frac{3}{4}\phi+\alpha\varphi} \right], \quad (3.33)$$

and satisfies the $N = 2$ relation (1.3).

Once again, it is instructive to examine the decomposition of $D = 4$, $N = 8$ (for the round S^7) to $D = 4$, $N = 2$ supersymmetry. In this case, $N = 2$ is preserved only for left-squashing, where $8_s \rightarrow 6_0 + 1_2 + 1_{-2}$ under $SO(8) \supset SU(4) \times U(1)$. Truncating to $SU(4)$ singlets, the massless ($n = 0$) Kaluza-Klein sector yields the pure $N = 2$ supergravity multiplet

$$\mathcal{D}(2, 1; 0) = D(3, 2; 0) + D(\frac{5}{2}, \frac{3}{2}; -1) + D(\frac{5}{2}, \frac{3}{2}; 1) + D(2, 1; 0). \quad (3.34)$$

In contrast to the five-dimensional case, no states at the $n = 1$ level survive the truncation. The breathing mode finally shows up at the second Kaluza-Klein level, where the truncation to $N = 2$ yields a massive vector

$$\begin{aligned} \mathcal{D}(4, 0; 0) &= D(5, 1; 0) \\ &+ D(\frac{9}{2}, \frac{1}{2}; -1) + D(\frac{9}{2}, \frac{1}{2}; 1) + D(\frac{11}{2}, \frac{1}{2}; -1) + D(\frac{11}{2}, \frac{1}{2}; 1) \\ &+ D(4, 0; 0) + D(5, 0; 0) + D(5, 0; -2) + D(5, 0; 2) + D(6, 0; 0). \end{aligned} \quad (3.35)$$

As before, the representations are given in terms of E_0 , spin and $U(1)$ charge. The breathing mode is identified as the $E_0 = 6$ scalar, while the squashing mode is the $E_0 = 4$ scalar. The remaining neutral (axionic) scalar may be identified with the three-form field strength $F_{[3]}$, which was set to zero in (3.19) but could as well have been retained [14].

4 Reduction of $D = 11$ Supergravity to seven Dimensions

The last case we consider is the reduction of eleven dimensional supergravity on S^4 . Since the even sphere cannot be written as a Hopf fibration, we cannot play the same trick to turn on a single squashing mode. Thus we focus only on the round S^4 .

For this case, the reduction ansatz is [14]

$$\begin{aligned} ds_{11}^2 &= e^{2\alpha\varphi} ds_7^2 + e^{2\beta\varphi} ds^2(S^4), \\ \hat{F}_{[4]} &= F_{[4]} + 6m\epsilon_{[4]}(S^4), \end{aligned} \quad (4.1)$$

with

$$\alpha = \frac{2}{3\sqrt{10}}, \quad \beta = -\frac{5}{4}\alpha, \quad (4.2)$$

whereupon the bosonic Lagrangian, (3.1), reduces to

$$e^{-1}\mathcal{L}_7 = R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2\cdot 4!}e^{-6\alpha\varphi}F_{[4]}^2 - 3me^{-1} * (F_{[4]} \wedge A_{[3]}) - V(\varphi), \quad (4.3)$$

where

$$V = 18m^2 e^{12\alpha\varphi} - R_4 e^{\frac{9}{2}\alpha\varphi}. \quad (4.4)$$

Note that, following [14], we have retained a possible four-form field strength, $F_{[4]}$, in seven dimensions. The potential has a minimum at

$$e^{\frac{15}{2}\alpha\varphi_*} = \frac{R_4}{48m^2}, \quad (4.5)$$

giving rise to an AdS₇ vacuum, provided $F_{[4]} = 0$.

In a 7 + 4 split, the Dirac matrices decompose as

$$\Gamma^M = (\gamma^\mu \otimes \tilde{\gamma}^5, 1 \otimes \tilde{\gamma}^a) \quad (4.6)$$

where $\tilde{\gamma}^5 = \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^4$ is the chirality operator on S^4 . In this case the $D = 11$ gravitino supersymmetry transformation, (3.2), reduces to

$$\begin{aligned} \delta \hat{\psi}_\mu &= \left[\nabla_\mu + \frac{\alpha}{2} \gamma_\mu{}^\nu \partial_\nu \varphi - \frac{m}{2} e^{6\alpha\varphi} \gamma_\mu - \frac{1}{12 \cdot 4!} e^{-3\alpha\varphi} (\gamma_\mu{}^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu \gamma^{\rho\lambda\sigma}) \tilde{\gamma}^5 F_{\nu\rho\lambda\sigma} \right] \hat{\epsilon}, \\ \delta \hat{\psi}_a &= \left[\tilde{\nabla}_a - \frac{5\alpha}{8} e^{-\frac{9}{4}\alpha\varphi} (\gamma \cdot \partial\varphi) \otimes \tilde{\gamma}_a \tilde{\gamma}^5 + m e^{\frac{15}{4}\alpha\varphi} \tilde{\gamma}_a \tilde{\gamma}^5 - \frac{1}{12 \cdot 4!} e^{-\frac{21}{4}\alpha\varphi} F_{[4]} \cdot \gamma \otimes \tilde{\gamma}_a \right] \hat{\epsilon}. \end{aligned} \quad (4.7)$$

For this case, the appropriate Killing spinors on S^4 satisfy

$$\left[\tilde{\nabla}_a \pm \frac{1}{2} \tilde{\gamma}_a \tilde{\gamma}^5 \sqrt{\frac{R_4}{12}} \right] \eta = 0, \quad (4.8)$$

yielding four Killing spinors η^i , $i = 1, 2, 3, 4$ and leading to the $D = 7$, $N = 4$ transformations

$$\begin{aligned} \delta \psi_{i\mu}^{(7)} &= \left[\nabla_\mu + \frac{1}{10} \left(3m e^{6\alpha\varphi} \mp 4 \sqrt{\frac{R_4}{12}} e^{\frac{9}{4}\alpha\varphi} \right) \gamma_\mu \right. \\ &\quad \left. - \frac{1}{480} e^{-3\alpha\varphi} (3\gamma_\mu{}^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu \gamma^{\rho\lambda\sigma}) \tilde{\gamma}^5 F_{\nu\rho\lambda\sigma} \right] \varepsilon_i^{(7)}, \\ \delta \lambda_i^{(7)} &= \frac{1}{2} \left[\gamma \cdot \partial\varphi - \frac{4}{5\alpha} \left(2m e^{6\alpha\varphi} \mp \sqrt{\frac{R_4}{12}} e^{\frac{9}{4}\alpha\varphi} \right) + \frac{1}{180\alpha} e^{-3\alpha\varphi} F_{[4]} \cdot \gamma \tilde{\gamma}^5 \right] \varepsilon_i^{(7)}. \end{aligned} \quad (4.9)$$

Here the seven-dimensional quantities are defined as

$$\begin{aligned} \varepsilon_i^{(7)} \otimes \eta^i &= e^{-\frac{1}{2}\alpha\varphi} \hat{\epsilon}, \\ \lambda_i^{(7)} \otimes \eta^i &= -\frac{1}{5\alpha} e^{\frac{7}{4}\alpha\varphi} \tilde{\gamma}^5 \tilde{\gamma}^a \hat{\psi}_a, \\ \psi_{i\mu}^{(7)} &= e^{-\frac{1}{2}\alpha\varphi} \hat{\psi}_\mu - \alpha \gamma_\mu \lambda_i^{(7)} \otimes \eta^i. \end{aligned} \quad (4.10)$$

The above equations can be written more suggestively as

$$\begin{aligned} \delta \psi_\mu &= \left[\nabla_\mu - \frac{1}{10\sqrt{2}} W \gamma_\mu - \frac{1}{480} e^{-3\alpha\varphi} (3\gamma_\mu{}^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu \gamma^{\rho\lambda\sigma}) \tilde{\gamma}^5 F_{\nu\rho\lambda\sigma} \right] \varepsilon, \\ \delta \lambda &= \frac{1}{2} \left[\gamma \cdot \varphi + \sqrt{2} \partial_\varphi W + \frac{1}{180\alpha} e^{-3\alpha\varphi} F_{[4]} \cdot \gamma \tilde{\gamma}^5 \right] \varepsilon, \end{aligned} \quad (4.11)$$

where the superpotential is identified as

$$W = -\sqrt{2} \left[3m e^{6\alpha\varphi} \mp 4 \sqrt{\frac{R_4}{12}} e^{\frac{9}{4}\alpha\varphi} \right]. \quad (4.12)$$

Note that this satisfies the identity (1.3) as expected.

5 Discussion

In the above, we have considered the supersymmetry of breathing mode compactifications for the three cases: $D = 10$ on S^5 , $D = 11$ on S^7 and $D = 11$ on S^4 . In all cases, the breathing mode is a singlet under the R symmetry, despite the fact that it lies in the massive Kaluza-Klein spectrum. For this reason, inclusion of the breathing mode in itself is allowed in a consistent truncation of the full Kaluza-Klein spectrum. However, for the resulting theory to be (maximally) supersymmetric, the superpartners to the breathing mode must also be retained. Presumably once these non-singlet superpartners are included, this would no longer be a consistent truncation unless the entire Kaluza-Klein tower is brought in as well. Thus it may not be entirely appropriate to regard this breathing mode compactification as solely a lower dimensional supergravity theory coupled to the breathing mode supermultiplet. While consideration of the higher dimensional equations of motion ensures the validity of the above reduction ansatz, the resulting theory of the form (1.1) and (1.2) is necessarily incomplete. The complete structure of the theory, and especially its supersymmetry, is perhaps more naturally seen in the original higher dimensional form.

On the other hand, for the squashed sphere compactifications, one may consistently truncate to the $SU(n+1)$ singlet sector of the full R symmetry group. Since this procedure already removes many states in the Kaluza-Klein tower, it suggests that the $N = 2$ truncations of the $D = 5$ and $D = 4$ theories may admit a further truncation yielding only the breathing/squashing multiplet coupled to the massless supergravity multiplet. If this were in fact the case, it would provide an interesting example of a consistent truncation to a massive supergravity theory where only a portion of the Kaluza-Klein tower survives.

Finally, note that recent investigation of the brane-world has shown that the ‘kinked’ Randall-Sundrum geometry is only compatible with supersymmetry provided the superpotential W changes sign when passing through the brane [1, 2, 38, 39, 40]. Restoring the gauge coupling constant g , this corresponds in the five-dimensional point of view to $g \rightarrow -g$ on opposite sides of the brane [39, 40]. This cannot be realized in a strictly $D = 5$, $N = 2$ point of view, but however is expected from the type IIB compactification, where it corresponds to a reversal of the orientation of S^5 . This orientation flip corresponds to making the opposite choice of sign in the Killing spinor equation on the sphere, (2.16). Furthermore, the five-form flux changes sign, $m \rightarrow -m$, so that the superpotential, (2.23) indeed flips sign as expected. This is also the case for the superpotential on the squashed S^5 , (2.55).

However, by changing the sign in the Killing spinor equation, (2.16), this orientation reversal joins opposite sets of Killing spinors on both sides of the Randall-Sundrum brane. This potential difficulty with supersymmetry is even more pronounced for the S^7 case where left- and right-squashing yield rather different realizations of the Killing spinors (cf. Table 3.2). Of course, the kinked brane-world is singular at the location of the brane. So perhaps it is not surprising to see this behavior of the Killing spinors upon orientation reversal. It remains to be seen what effect this has on a complete understanding of the supersymmetry of the brane-world.

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